USP

Some more mathematical formulation of stock dynamics

- Purpose of slides
  - Introduce the basics in mathematical representation of population dynamics in some detail
    - Stock production models
      - A revision
    - Cohort based models (generic length/age models)
      - An introduction
  - Show how the models are all a a special form of the mass balance equation (Russels equation)

- Source:
  - Haddon 2001: Chapter 1 & 2
  - Hilborn and Walters 1992: Chapter 3.4

Russel’s mass balance formulation

$$\begin{pmatrix}\text{next}\\ \text{biomass}\end{pmatrix}=\begin{pmatrix}\text{last}\\ \text{biomass}\end{pmatrix}+(\text{recruitment})+(\text{growth})-\begin{pmatrix}\text{natural}\\ \text{mortality}\end{pmatrix}-(\text{catch})$$

- Russels contribution:
  - “.. the sole value of the exact formulation given above is that it distinguishes the separate factors making up gain and loss respectively, and is therefore an aid to clear thinking” (Russel 1931)
  - Recognized that a stock could be divided into animals that were in the fishable stock and those that were entering the fishable stock at any one time (recruitment)
  - Stock biomass has gains: Recruitment and growth
  - Stock biomass has losses: Natural and fishing mortality (catch)

Russel's equation: A mass balance equation

- $B_{y+1} = B_y + R_y + G_y - M_y - Y_y$

  - B_{y+1} stock size in weight at start of time y+1
  - B_y stock size in weight at start of time y
  - R_y weight of all recruits entering stock at time y
    - Recruits: Young fish "entering" the stock in each time period
  - G_y weight gain of fish surviving from y to y+1
  - M_y weight loss of fish that died from y to y+1
  - Y_y weight of fish captured from y to y+1

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Stock production models

Purpose of slides

- Introduce the basics in mathematical representation of population dynamics in some detail:
  - Show how the models are all a a special form of the mass balance equation (Russels equation)
  - Stock production models
  - Cohort based models (generic length/age models)
    - Describe the general pattern observed
    - Derive the mathematical equation

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  - Hilborn and Walters 1992: Chapter 3.4

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Russel's equation modified

- Russel's equation for biomass:

$$B_{y+1} = B_y + R_y + G_y - M_y - Y_y$$

- The two sources of increase are called production:

$$\text{Production} = (R_y + G_y)$$

- Difference between production and natural mortality is called surplus production:

$$\text{Surplus production} = sP_y = (R_y + G_y) - M_y$$

- Thus have:

$$B_{y+1} = B_y + sP_y - Y_y$$

- Assume that surplus production is function of the current biomass, i.e.

$$sP_y = f(B_y)$$

- Thus:

$$B_{y+1} = B_y + f(B_y) - Y_y$$

Functional forms of surplus productions

- Classic Schaefer (logistic) form:

$$f\left(B_y\right) = rB_y\left(1 - \frac{B_y}{K}\right)$$

- The more general Pella & Tomlinson form:

$$f\left(B_y\right) = \frac{r}{p}B_y\left(1 - \left[\frac{B_y}{K}\right]^p\right)$$

  - Note: when p=1 the two functional forms are the same
  - If p <> 1, then the density dependence is no longer linear

Production model: Data needed

- The model in math

$$B_{y+1} = B_y + r\left(\frac{K - B_y}{K}\right)B_y - Y_y$$

$$CPUE_y = qB_y$$

- Need a <u>time</u> <u>series</u> of:
  - Total annual catch (Y_y)
  - Index of abundance ($CPUE_y$)
- What we get out:

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Cohort models

Russel's equation modified 1

- Russel's equation for <u>biomass:</u>

$$B_{y+1} = B_y + R_y + G_y - M_y - Y_y$$

- Russel's equation for <u>numbers</u>:

$$N_{y+1} = N_y + R_y - D_y - C_y$$

  - N_{y+1} stock size in numbers at start of time y+1
  - N_y stock size in numbers at start of time y
  - R_y number of recruits entering stock at time y
  - D_y number of fish that died from y to y+1
  - C_y number of fish that are caught from y to y+1

    - What happened to the growth term??

Russel's equation modified 2

- Russel equation for <u>numbers</u>:

$$N_{y+1} = N_y + R_y - D_y - C_y$$

- If we are considering <u>ONE</u> cohort only we can drop the recruitment term and have

$$N_{y+1} = N_y - D_y - C_y$$

- If we assume that the total number dying of natural causes (D_y) are a proportion of those living i.e.:

$$D_y = f(N_y) = mN_y$$

  - m: proportion of fish that die of natural causes during time interval y to y+1

- We thus have

$$N_{y+1} = N_y - m_y N_y - C_y$$

Russel's equation modified 3

- From last slide we have

$$N_{y+1} = N_y - m_y N_y - C_y$$

- We could think that the proportion that removed by fishing is also a fraction of the population size, i.e.

$$C_y = f_y N_y$$

- And thus:

$$N_{y+1} = N_y - m_y N_y - f_y N_y$$

$$N_{y+1} = N_y (1 - (m_y + f_y))$$

In a biological sense this equation is actually wrong, since here we first let all fish die of natural causes before we subtract the fish that are removed by fishing. However, in most fisheries both events occur continuously over the time period between y and y+1 and thus not mutually exclusive events.

Russel's equation modified 4: The stock equation

- To correct for the events not being mutually exclusive we could do this:

$$N_{y+1} = N_y \left(1 - (m_y + f_y - m_y f_y)\right)$$

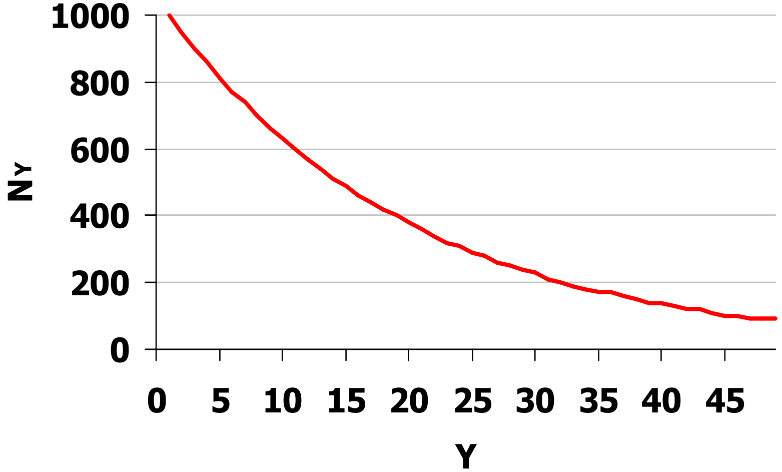
  This mathematical still awkward. A much easier mathematical form, albeit less comprehensible form is:

$$\mathrm{N}_{\mathrm{y+1}} = N_y e^{-\left(M_y + F_y\right)}$$

- The difference between the small capital and the large capital letters is:
  - Small capital m and f: an annual proportion
  - Large capital M and F: an instantaneous rate

After birth there is only death!

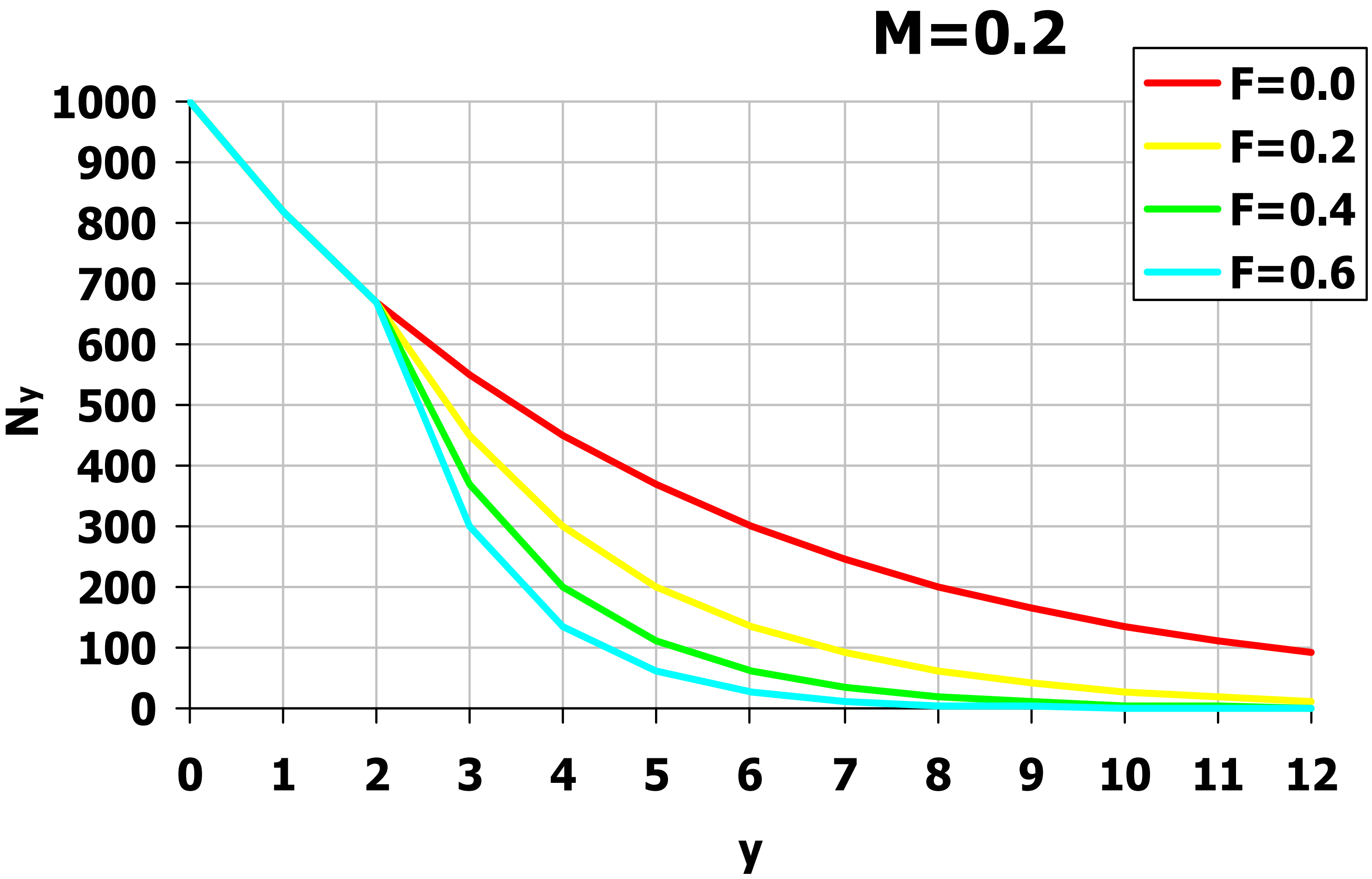
$$N_{y+1} = N_y e^{-\left(M_y + F_y\right)} = N_y e^{-Z_y}$$



- After hatching the individuals in a cohort can only decline
- If this does not hold true the stock is wrongly defined

The effect of fishing

$$N_{y+1} = N_y e^{-\left(M_y + F_y\right)}$$



The stock equation in full

- If we were only dealing with a single cohort, then mathematically the following is sufficient

$$\mathrm{N}_{\mathrm{y+1}} = N_y e^{-\left(M_y + F_y\right)}$$

- However, because the stock in a particular year is composed of many cohorts we need to make a mathematical distinction between age groups:

$$\mathrm{N}_{\mathrm{a+1,y+1}} = N_{a,y} e^{-\left(M_{a,y} + F_{a,y}\right)}$$

Describing the catch 1

- The number of fish that die in each time interval is:

$$D_y = N_y - N_{y+1}$$

- Substituting with the stock equation we get:

$$D_y = N_y - N_y e^{-\left(M_y + F_y\right)}$$

$$= N_y \left(1 - e^{-\left(M_y + F_y\right)}\right)$$

$$\begin{pmatrix} \text{Numer of} \\ \text{fish that die} \\ \text{during the time} \end{pmatrix} = \begin{pmatrix} \text{Number of} \\ \text{fish alive in} \\ \text{the beginning} \end{pmatrix} \begin{pmatrix} \text{Proportion of} \\ \text{fish that die} \end{pmatrix}$$

Describing the catch 2

- The number that die due to fishing mortality is the fraction of the number of fish that die (F_y/Z_y), i.e.

$$C_y = \frac{F_y}{F_y + M_y}\left(1 - e^{-\left(F_y + M_y\right)}\right)N_y$$

$$\begin{pmatrix}\text{Numer of} \\ \text{fish fished} \\ \text{during the time}\end{pmatrix} = \begin{pmatrix}\text{Proportion of} \\ \text{fish that die} \\ \text{due to fishing}\end{pmatrix}\begin{pmatrix}\text{Proportion of} \\ \text{fish that die}\end{pmatrix}\begin{pmatrix}\text{Number of} \\ \text{fish alive in} \\ \text{the beginning}\end{pmatrix}$$

  - C_y: The number of fish caught during the year y

Describing the catch 3: The catch equation

- Because the catch in a particular year is composed of many cohorts we need to make a mathematical distinction between age groups:

$$C_{a,y} = \frac{F_{a,y}}{F_{a,y} + M_{a,y}} \left(1 - e^{-\left(F_{a,y} + M_{a,y}\right)}\right) N_{a,y}$$

  - C_{ay}: The number of fish at age a caught over time the year y

The "link model"

- In addition to catch data we have relative measure of stock abundance (U_{ay}).
  - Normally assume:

$$U_{a,y} = q_a N_{a,y}$$

  - However often also seem to have:

$$U_{a,y} = q_a N_{a,y}^{\beta_a}$$

Assessment methods in brief

- We want to estimate:
  - a matrix of Nay values
  - a matrix of Fay values
- from:
  - a matrix of $C_{a,y}$ measurements
  - a matrix of $U_{a,y}$ measurements
- In order to do so we need to have some estimates of $M_{a,y}$. These are however not normally available, thus often assume that natural mortality is constant.

Another way of putting it:

- We have:
  - Relative measures of stock abundance (surveys)
  - Absolute measures of removals from the stock (catches)
- We want:
  - Absolute numbers of stock abundance (stock numbers)
  - Removals relative to stock abundance (mortalities)

Age/size structured models

- Advantages
  - Populations do have age/size structure
  - Basic biological processes are age/size specific
    - Growth
    - Mortality
    - Fecundity
  - The process of fishing is age/size specific
  - Relatively simple to construct mathematically
  - Model assumption not as strict as in e.g. production models
- Disadvantages
  - Sample intensive
    - Data often not available
    - Mostly limited to areas where species diversity is low
  - Have to have knowledge of natural mortality
  - For long term management strategies have to make model assumptions about the relationship between stock and recruitment
  - Often not needed to address the question at hand